Title Representing Matroids using Partial Fields

Author Daniel Slilaty, joint work with Jakayla Robbins

Abstract Matroids were developed independently in the 1930's by Hassler Whitney and Takeo Nakasawa. A matroid is an abstraction of the properties of linear independence among collections of vectors. Formally, the definition is as follows. A *matroid* M is a finite set E along with a collection $\mathcal{I}$ of subsets of E called *independent* such that:

(1) $\mathcal{I}$ is closed under taking subsets and

(2) whenever $I, J \in \mathcal{I}$ with $|I| < |J|$, there is $e \in J - I$ such that $I \cup e \in \mathcal{I}$.

Of course, the canonical example of a "matroid" is given by a collection of vectors E being the columns of some "matrix" A over a field $\mathbb{F}$. The independent sets $\mathcal{I}$ is the collection of linearly independent subsets of E. When matroid M has a matrix representation A over some field $\mathbb{F}$, we call M $\mathbb{F}$-*representable*.

A *partial field* is a relatively new notion in the field of matroid representations. It is a pair $\mathbb{P} = (R, G)$ in which R is a commutative unitary ring along with a subgroup G of the group of units of R such that $-1 \in G$. A $\mathbb{P}$-matrix is a matrix A over R for which every non-zero sub-determinant of A is in G. The most famous example of such is the class of *totally unimodular matrices*: $\mathbb{Z}$-matrices for which every non-zero sub-determinant is in $\{+1, -1\}$. (Totally unimodular matrices are of central importantance in the field of linear optimization as shown by Kruskal and Hoffman in 1956.)

Theorem 1 is an early result in matroid theory which has inspired a variety of analogous results for $GF(3)$-representable matroids by Whittle and $GF(4)$- and $GF(5)$-representable matroids by Vertigan as well as Pendavingh and Van Zwam.

Theorem 1 (Tutte)**.** *If M is a binary matroid, then M has a totally unimodular representation if and only if M is $\mathbb{F}$-representable for some field $\mathbb{F}$ with char$(\mathbb{F}) \neq 2$.*

Theorem 1 and its inspired results all concern the idea that a matroid representable over some collection of different fields should be representable over some appropriate partial field. A similar but less-explored line of inquiry concerns the notion that a matroid that is both representable over some finite field and *orientable* should also have a representation over some appropriate partial field of real numbers. (An *orientation* of a matroid is a generalization of linear-independence properties for collections of $\mathbb{R}$-valued vectors.) The first such result is by Bland and Las Vergnas in their paper which first introduced oriented matroids.

Theorem 2 (Bland and Las Vergnas)**.** *If M is a binary matroid, then $\mathcal{O}$ is an orientation of M if and only if $\mathcal{O}$ is the orientation induced by a totally unimodular representation of M.*

Lee and Scobee discovered the analogue to Theorem 2 for ternary matroids. Theorem 3 has been the extent of knowledge concerning partial-field representations of matroids that are both orientable and representable over some finite field. The *dyadic* partial field is $\mathbb{D} = (\mathbb{Q}, \langle -1, 2 \rangle)$.

Theorem 3 (Lee and Scobee)**.** *If M is a ternary matroid, then $\mathcal{O}$ is an orientation of M if and only if $\mathcal{O}$ is the orientation induced by a dyadic representation of M.*

In this talk, we will address the situation for orientations of quaternary (i.e., $GF(4)$-representable) matroids that are induced by golden-mean representations. The *golden-mean* partial field is $\mathbb{G} = (\mathbb{R}, \langle -1, g \rangle)$ in which $g = \frac{1+\sqrt{5}}{2}$, the *golden ratio*. While it is not true that every orientation of a quaternary matroid M is induced by a golden-mean representation of M (e.g., $U_{3,6}$ has 372 orientations and only 12 of them are golden-mean orientations) there is an easily identified subset of them that are. We will define what it means for an orientation of a quaternary matroid to be *consistently ordered* and note that it is an obvious necessary condition for a $GF(4)$-representation and orientation to be induced by a golden-mean representation.

Theorem 4 (Robbins and Slilaty)**.** *If M is a quaternary matroid, then $\mathcal{O}$ is a consistently ordered orientation of M if and only if $\mathcal{O}$ is the orientation induced by a golden-mean representation of M.*

The proof of Theorem 4 relies on Tutte's Homotopy Theorem as well as computer verification that the result is true for rank-3 matroids.